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Electron acceleration in combined laser and uniform electric fields

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Abstract

An exact analytical solution for the equation of motion of a single relativistic electron, injected initially at some angle ξ to the propagation direction of a linearly polarized plane-wave laser field of arbitrary intensity and a uniform electric field oriented anti-parallel to the laser propagation direction, is developed. The solution is then used to investigate the issue of electron acceleration to high energies in the prescribed fields. It is found that, in principle, an electron may be accelerated from rest or motion to several hundred GeV, if the uniform electric field strength E_s approaches a critical value $E_s^c = (m\tilde{\omega}c)/(2\pi Ne)$, where m and e are the mass and charge of the electron, c is the speed of light in vacuum, N is the number of field cycles in the pulse and $\tilde{\omega}$ is the Doppler-shifted frequency of the laser field as seen by the electron upon initial injection. The radiation losses during acceleration are shown to be negligible and the spectrum of the radiation emitted along the initial direction of motion (parallel injection) of the electron is shown to consist mostly of the fundamental laser frequency.

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(Some figures in this article are in colour only in the electronic version; see `www.iop.org`)

1. Introduction

Theoretical studies of such laser-assisted atomic and molecular processes as high-harmonic generation and negative-ion photodetachment have shown that the electron moves, for part of the time, as a free particle driven by the laser field (see, e.g., [1]). The electron exchanges a lot of energy with the laser field and as a result is accelerated, especially when relativistic laser intensities are involved [2], and in the regime in which it is free from all binding effects. The

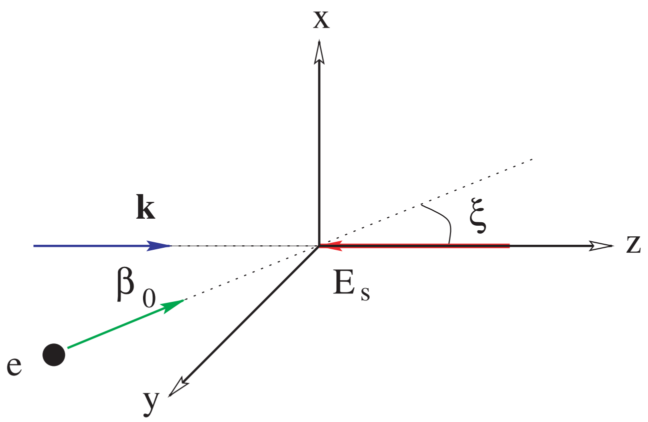

Figure 1. A schematic diagram showing the configuration of electron injection into the electromagnetic fields.

added presence of a static electric and/or magnetic field, used to control such processes [3–6], can also alter the electron dynamics quite drastically.

This continues to motivate efforts to investigate problems concerning the interaction of a single free electron with ultra-high-intensity laser and other electromagnetic fields [7–19]. This paper is about an electron injected at an angle to the direction of propagation of a plane-wave linearly polarized intense laser field. In addition, a uniform static electric field is oriented anti-parallel to the laser propagation direction (see figure 1 for a schematic diagram). The study is further motivated by the continued interest in the related issues of particle acceleration to extremely high energies [20, 21]. Recent advances in laser technology [22, 23], especially with regard to the production of very high-intensity pulses containing a small number of field cycles, seem to justify the renewed interest in these problems [24, 25].

It is well known that the electron gains energy, and is thus accelerated, continuously in the presence of the static electric field acting alone on it; this is the basic idea utilized in conventional accelerators. It is also known that, in a plane-wave laser field alone, the electron gains virtually no net energy as a result of interaction with an integer number of laser field cycles [8, 26, 27]. A net gain is possible if the electron is extracted after it has interacted with a non-integer number of field cycles, with this gain reaching a maximum if extraction takes place at the end of an odd-integer number of half-cycles of the radiation field. Limitations on the highest laser field intensities available in the past have placed severe limits on the utility of electromagnetic waves in accelerating electrons. Recent theoretical work based on numerical simulation methods has concluded that an electron may be captured and violently accelerated when interacting with an ultra-high-intensity Hermite–Gaussian laser beam [28, 29]. Equally, the presence of a nucleus during ionization or scattering could trigger the extraction of energy from the laser field [30]. On the other hand, the addition of an extra applied electric or magnetic field has been shown [31–40] to destroy the symmetry of the plane-wave field and to allow for efficient particle acceleration. These ideas have been utilized in the construction of several types of accelerator including inverse Cerenkov accelerators [41], inverse free-electron laser accelerators [42] and plasma laser accelerators [43, 44]. Electron energy gradients of up to 50 GeV m^{-1} have recently been reported in self-modulated plasma-wakefield laser accelerators [45–48].

In this paper, our in-principle calculations demonstrate that the addition of a weak dc electric field (weak by comparison to the electric component of the laser field employed) does not significantly alter the electron trajectory. However, when the added electric field strength is increased the phase of the radiation field, which will be employed as a convenient variable in all our calculations, is distorted. High energy gains by the electron become possible as the electric field strength E_{s} approaches the critical value

$$E_{\mathrm{s}}^{\mathrm{c}} = \frac{m\omega_{\mathrm{l}} c}{2\pi N e}\gamma_0(1 - \beta_0 \cos\xi). \tag{1}$$

In equation (1) m and e are the mass and charge of the electron, ω_{l} and N are the frequency and number of laser field cycles, $\gamma_0 = (1 - \beta_0^2)^{-1/2}$ and β_0 is the initial speed of the electron scaled by the speed of light. It will be shown that for values of $E_{\mathrm{s}} \to E_{\mathrm{s}}^{\mathrm{c}}$ the forward electron speed will be close to the phase velocity $v_{\mathrm{ph}} \approx c$ of the accelerating electric field (that of the laser). This will keep phase-slippage to a minimum and the electron will absorb energy from the radiation field continuously. We show in this paper that in this regime electron energies of up to a few hundred GeV may be reached by electrons starting from rest at the origin of coordinates. Less energy gain per metre of forward travel will be shown to be possible for electrons injected with nonzero initial forward momentum. Unfortunately, the required dc electric field strengths are high and need to be maintained over long distances. An acceleration scheme based on this idea with present-day technology may not be feasible. However, one way a future design may solve the problem of the dc field is by building the long field environment from many successive small cells. As an example, let an electric field of strength $E_{\mathrm{s}} = 5.1 \times 10^8$ V m^{-1} act alone on a 4.5 MeV electron ($\gamma_0 = 10$, injection energy= $\gamma_0 mc^2$) injected antiparallel to it. Such a field accelerates the electron to about 52 GeV over 100 m. On the other hand, when a plane-wave laser field of intensity $\approx 10^{20}$ W cm^{-2} acts alone on the same electron, the latter reaches a maximum energy of approximately 2 GeV. However, when both fields act concurrently, in the fashion described above, electron energies of up to 120 GeV may be reached over a distance of 100 m.

When the electron undergoes acceleration, it emits radiation and therefore loses part of its net energy gain. This so-called *radiation loss* places a limit on the highest energy attainable by the electron. On the other hand, the radiation can be very useful, especially if its spectrum contains high harmonics of the incident radiation frequency. We will show that the radiation losses during the acceleration process are negligible compared to the net energy gain. The frequency distribution of the radiation emitted along the initial direction of motion of the electron will also be discussed.

Besides the requirement that a high dc electric field be maintained over a long distance, the laser field intensities required are high. Present-day high intensities are produced by focusing over small dimensions, of the order of a few micrometres. An accelerator design based on our equations requires that the high intensity be maintained over a distance of typically many metres. This is also not feasible with today's technology. Obviously the need arises for the construction of optical elements that would not only withstand the high field intensities we are talking about, but would also produce multiple focusing of the beam over long distances.

In broad outline, our approach consists of solving the classical relativistic equation of motion of the electron in the electromagnetic field environment described above exactly analytically. Radiation reaction effects will be ignored as recent studies [49] have shown that these are rather small even for laser field intensities as high as 10^{22} W cm^{-2}. Analytic solutions of the relativistic equation of motion in electromagnetic plane waves go back many years [27, 32, 50–55], but similar solutions in the case of an added dc field are rare. In fact, to the best of our knowledge, the case of an added dc electric field has never been investigated exactly analytically before. However, solutions to the Klein–Gordon and Dirac equations for a particle in such an environment were found many years ago [56] and an underlying classical solution to the problem at hand may, in principle, be obtained from those solutions in the appropriate limits.

The results we present below include exact expressions for the particle energy, velocity components and trajectory in terms of the phase of the laser as a convenient parameter. Subsequently, the energy expression is studied in depth and the condition under which it may be maximized is investigated. This is of paramount importance for the design and operation of novel particle accelerators employing the strong laser fields that are currently available for laboratory experiments. On the other hand, the velocity and trajectory expressions are central for the calculation of the radiation spectra that result from the re-emission of part of the energy the electron absorbs from the radiation field.

The main working equations will be derived in section 2. It will be shown in the same section that the basic equations have the expected limits when the static electric field vanishes and also in the absence of the laser field. The issue of particle acceleration will be taken up in section 3, where the most appropriate condition for energy gain is arrived at and the accompanying radiation losses are discussed. A brief, but general, plan for calculating the emission spectra will be laid out in section 4 and will be illustrated by the simplest example, namely the one pertaining to an observation direction along the initial direction of motion of the electron. We end by giving a short discussion and a brief summary of our main conclusions in section 5.

2. The basic working equations

2.1. Derivation

We focus attention on the motion of an electron, of mass m, charge $-e$ and energy–momentum four-vector $p = (\mathcal{E}/c, \boldsymbol{p})$, where

$$\mathcal{E} = \gamma mc^2 \qquad \boldsymbol{p} = \gamma mc\boldsymbol{\beta} \tag{2}$$

$\gamma = (1-\beta^2)^{-\frac{1}{2}}$ is the Lorentz factor and $\boldsymbol{\beta}$ is the electron velocity normalized by c, the speed of light. Let a uniform electric field of strength E_s be oriented anti-parallel to the direction of propagation of an intense plane-wave laser field of (vector potential) amplitude A_l and frequency ω_l polarized in the $+x$ direction. Suppose the electron is injected at time $t = 0$ with speed $v_0 = c\beta_0$ in a direction making the angle ξ with that of propagation of the laser field. The combined uniform electric and laser fields may be described in a unified fashion by the vector potential

$$\boldsymbol{A} = \hat{\boldsymbol{\imath}} A_l \cos\eta + \hat{\boldsymbol{k}} cE_s t \tag{3}$$

where $\eta = \omega_l t - \boldsymbol{k}\cdot\boldsymbol{r}$, $\boldsymbol{k}$ is the laser propagation vector taken along the $+z$ direction and t and $\boldsymbol{r}$ are the time and space coordinates of the electron. Furthermore, $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{k}}$ are unit vectors in the coordinate $+x$ and $+z$ directions, respectively. As usual, the electric and magnetic fields will be derived from the vector potential $\boldsymbol{A}$ via the equations

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \qquad \boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}. \tag{4}$$

In this paper, the relativistic equation of motion will be solved exactly analytically. The said equation may be broken up into the following vector and scalar parts:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -e(\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B}) \tag{5}$$

and

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -ec\boldsymbol{\beta}\cdot\boldsymbol{E}. \tag{6}$$

In component form, equations (5) and (6) are equivalent to the following set:

$$\frac{\mathrm{d}(\gamma\beta_x)}{\mathrm{d}t} = -q\omega_1(1-\beta_z)\sin\eta \tag{7}$$

$$\frac{\mathrm{d}(\gamma\beta_y)}{\mathrm{d}t} = 0 \tag{8}$$

$$\frac{\mathrm{d}(\gamma\beta_z)}{\mathrm{d}t} = -q\omega_1\beta_x\sin\eta + \left(\frac{eE_\mathrm{s}}{mc}\right) \tag{9}$$

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = -q\omega_1\beta_x\sin\eta + \left(\frac{eE_\mathrm{s}}{mc}\right)\beta_z \tag{10}$$

where the dimensionless laser intensity parameter $q = eA_1/mc^2$ has been introduced, with $q^2 = 1$ being equivalent to the field intensity $\approx 10^{18}$ W cm^{-2}.

Before any attempt is made at integrating this set of equations, it will be very helpful to present a few identities first. We will briefly sketch the derivation of some of those identities here. To begin with, direct differentiation of the phase with respect to the time variable gives

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \omega_1(1-\beta_z). \tag{11}$$

Now subtracting equation (9) from (10) and using the identity (11) in the result gives the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\gamma(1-\beta_z)\right] = -\left(\frac{eE_\mathrm{s}}{mc\omega_1}\right)\frac{\mathrm{d}\eta}{\mathrm{d}t}. \tag{12}$$

Subsequent integration of equation (12) finally yields the second identity

$$\gamma(1-\beta_z) = \gamma_0(1-\beta_{z0}) - \left(\frac{eE_\mathrm{s}}{mc\omega_1}\right)(\eta-\eta_0). \tag{13}$$

Note that the subscript 0 signifies an initial value, at $t = 0$, for the quantity in question. In the absence of E_s, equation (13) becomes a statement of conservation of the quantity $\mathcal{E}/c - p_z$, where $p_z = \gamma mc\beta_z$ is the electron's forward momentum. With E_s assuming a finite value, this quantity is no longer a constant of the motion.

Now with Q standing for x, y or z, direct differentiation and subsequent use of equation (13) lead easily to the third identity

$$\frac{\mathrm{d}Q}{\mathrm{d}\eta} = \frac{c}{\omega_1}\frac{\gamma\beta_Q}{\left[\gamma_0(1-\beta_{z0}) - \left(\frac{eE_\mathrm{s}}{mc\omega_1}\right)(\eta-\eta_0)\right]}. \tag{14}$$

In this paper, we are interested in the dynamics of a single electron in the simultaneous presence of a finite-duration laser pulse and a uniform electric field. A set of *initial conditions* on the electron's position and velocity, which we think are appropriate to the situation at hand, may be expressed compactly as

$$\boldsymbol{r}_0 = 0 \qquad \boldsymbol{\beta}_0 = \beta_0(\hat{\boldsymbol{\imath}}\sin\xi + \hat{\boldsymbol{k}}\cos\xi). \tag{15}$$

In other words, our electron is assumed to be moving uniformly in the xz plane, with the speed $v_0 = c\beta_0$ along a direction making the angle ξ with the coordinate $+z$-axis, when it is overtaken by the front edge of the pulse at the origin, the pulse being a plane wave with sharp turn-on and off. Note that our choice of initial conditions necessarily implies that $\eta_0 = 0$. Subject to these initial conditions, equations (7) and (8) may be integrated immediately, with the results

$$\gamma\beta_x = q(\cos\eta - 1) + \gamma_0\beta_0\sin\xi \tag{16}$$

$$\gamma\beta_y = 0. \tag{17}$$

Using these results in equation (14) and carrying out the integration over η, subject to the initial conditions expressed in equation (15), leads to

$$x(\eta) = \frac{c}{\omega_l}\left(\frac{\omega_l A_l}{cE_s}\right) I(\eta) \tag{18}$$

$$y(\eta) = 0. \tag{19}$$

In equation (18)

$$\begin{aligned} I(\eta) = \int_0^\eta & \frac{\left[1 - (\gamma_0\beta_0/q)\sin\xi - \cos\eta'\right]}{\eta' - \alpha}\,\mathrm{d}\eta' \\ = & \cos\alpha\,[\mathrm{Ci}(\alpha) - \mathrm{Ci}(\alpha - \eta)] + \sin\alpha\,[\mathrm{Si}(\alpha) - \mathrm{Si}(\alpha - \eta)] \\ & + \left[1 - \left(\frac{\gamma_0\beta_0}{q}\right)\sin\xi\right]\ln\left(1 - \frac{\eta}{\alpha}\right) \end{aligned} \tag{20}$$

and

$$\alpha = \left(\frac{mc\omega_l}{eE_s}\right)\gamma_0(1 - \beta_0\cos\xi) \tag{21}$$

and Si and Ci stand for the sine and cosine integral functions. Recall that the strength of the electric component of the laser field is given by

$$E_l = \frac{\omega_l}{c} A_l. \tag{22}$$

In light of this we see immediately that, according to equations (18)–(21), the transverse motion of the electron is mainly governed by the ratio E_l/E_s. Note also that $I(\eta)$ is only sensitive to changes in E_s alone.

Next we replace β_x and β_z on the right-hand sides of equations (9) and (10) by

$$\beta_x = \frac{1}{c}\frac{\mathrm{d}x}{\mathrm{d}\eta}\frac{\mathrm{d}\eta}{\mathrm{d}t} \qquad \beta_z = \frac{1}{c}\frac{\mathrm{d}z}{\mathrm{d}\eta}\frac{\mathrm{d}\eta}{\mathrm{d}t} \tag{23}$$

and use equation (11) in the results. When the remaining integrations are finally carried out, we obtain

$$\gamma(\eta) = \gamma_0 - q\left(\frac{\omega_l A_l}{cE_s}\right) J(\eta) + \left(\frac{eE_s}{mc^2}\right) z(\eta) \tag{24}$$

$$\left[\gamma\beta_z\right](\eta) = \gamma_0\beta_0\cos\xi - q\left(\frac{\omega_l A_l}{cE_s}\right) J(\eta) + \left(\frac{eE_s}{mc\omega_l}\right)\left[\eta + \frac{\omega_l}{c} z(\eta)\right] \tag{25}$$

where

$$\begin{aligned} J(\eta) = \int_0^\eta & \frac{\sin\eta'\left[1 - (\gamma_0\beta_0/q)\sin\xi - \cos\eta'\right]}{\eta' - \alpha}\,\mathrm{d}\eta' \\ = & \left[1 - \left(\frac{\gamma_0\beta_0}{q}\right)\sin\xi\right]\{\sin\alpha\,[\mathrm{Ci}(\alpha - \eta) - \mathrm{Ci}(\alpha)] \\ & + \cos\alpha\,[\mathrm{Si}(\alpha) - \mathrm{Si}(\alpha - \eta)]\} - \tfrac{1}{2}\sin(2\alpha)\,[\mathrm{Ci}(2\alpha - 2\eta) - \mathrm{Ci}(2\alpha)] \\ & - \tfrac{1}{2}\cos(2\alpha)\,[\mathrm{Si}(2\alpha) - \mathrm{Si}(2\alpha - 2\eta)]\,. \end{aligned} \tag{26}$$

Note at this point as well, for future reference, that the value of $J(\eta)$ is sensitive to changes in E_s but not E_l. Now, using equation (25) in (14) for $Q = z$ formally yields

$$z(\eta) = -\frac{c}{\omega_l}\left(\frac{mc\omega_l}{eE_s}\right)\int_0^\eta \frac{[\gamma\beta_z](\eta')}{\eta' - \alpha}\,\mathrm{d}\eta'. \tag{27}$$

Equations (25) and (27) are coupled. To uncouple them, we add to the set of identities derived above yet another one, obtained by multiplying equation (10) by γ and subtracting from the result equation (9) multiplied by $\gamma\beta_z$

$$\gamma\frac{\mathrm{d}\gamma}{\mathrm{d}t} - \gamma\beta_z\frac{\mathrm{d}(\gamma\beta_z)}{\mathrm{d}t} = -q\omega_l \sin\eta(\gamma\beta_x)(1-\beta_z) \tag{28}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\gamma^2 - (\gamma\beta_z)^2\right] = -q^2\sin\eta\left[\cos\eta - 1 + \left(\frac{\gamma_0\beta_0}{q}\right)\sin\xi\right]\frac{\mathrm{d}\eta}{\mathrm{d}t} \tag{29}$$

where equations (11) and (16) have been used to obtain equation (29). A single integration, subject to the above-mentioned initial conditions, finally results in the identity

$$\gamma^2 - (\gamma\beta_z)^2 = \gamma_0^2 - (\gamma_0\beta_0\cos\xi)^2 + q^2(\cos\eta - 1)\left[\cos\eta - 1 + 2\left(\frac{\gamma_0\beta_0}{q}\right)\sin\xi\right]. \tag{30}$$

To complete the decoupling of equations (25) and (27) we evaluate the left-hand side of equation (30) using equations (24) and (25) and solve the result for $z(\eta)$. Thus we obtain

$$z(\eta) = \frac{c}{\omega_l}\left\{\left(\frac{\omega_l A_l}{cE_s}\right)^2 J(\eta) + \frac{1}{2\alpha}\left(\frac{\omega_l A_l}{cE_s}\right)^2\frac{(\cos\eta - 1)\left[\cos\eta - 1 + 2\left(\frac{\gamma_0\beta_0}{q}\right)\sin\xi\right]}{1-\eta/\alpha}\right.$$
$$\left. + \frac{1}{2}\left[\frac{\eta}{\alpha} + \frac{2\beta_0\cos\xi}{1-\beta_0\cos\xi}\right]\left(\frac{\eta}{1-\eta/\alpha}\right)\right\}. \tag{31}$$

Alternatively, the same expression for $z(\eta)$ may be arrived at directly from

$$\frac{1}{\gamma^2} = 1 - (\beta_x^2 + \beta_z^2) \tag{32}$$

together with equations (16), (24) and (25). Note that equation (19) implies that $\beta_y = 0$ at all times. Without an explicit knowledge of the integral $J(\eta)$ it is not obvious what effect the field E_s has on the electron dynamics. This issue will be elucidated in the next subsection.

We conclude this subsection by deriving a general expression for the electron energy, scaled by its rest energy mc^2. When equation (31) is used in equation (24), we get the following expression for the scaled energy of the particle:

$$\gamma(\eta) = \gamma_0\left\{1 + \frac{s(\eta)}{\left[1-\frac{\eta}{\alpha}\right]}\right\} \tag{33}$$

where

$$s(\eta) = \frac{1}{2}(1-\beta_0\cos\xi)\left(\frac{\eta}{\alpha}\right)^2 + \beta_0\cos\xi\left(\frac{\eta}{\alpha}\right) + \frac{q^2/2}{\gamma_0^2(1-\beta_0\cos\xi)}$$
$$\times(\cos\eta - 1)\left[\cos\eta - 1 + 2\left(\frac{\gamma_0\beta_0}{q}\right)\sin\xi\right]. \tag{34}$$

It is interesting to note that the dependence upon the integral $J(\eta)$ has dropped out of $\gamma(\eta)$ completely. This makes the scaled energy expression easier to interpret and manipulate for the sake of drawing conclusions about the exchange of energy between the electron and the laser field.

Now, equations (18), (19), (31) and (33) serve as the basis for a complete description of the dynamics of the electron in the combined presence of laser and uniform electric fields. The equations have been written in such a way as to show the competing roles played by these fields. Besides, the way they have been written makes checking the units and dimensions in those equations straightforward. In the next two subsections some important limits will be considered.

2.2. Limit of small E_s

We show now that, in the special case of forward injection ($\xi = 0$), the trajectory and energy equations have the correct limits for a vanishingly small uniform electric field. Note first that for a small E_s value the parameter α is very large, in which case we employ a Taylor series expansion in η/α in the integrals I and J and carry out the integrations term by term. Thus, setting $\xi = 0$, we have

$$I(\eta) = -\frac{1}{\alpha}\int_0^\eta \left(1 + \frac{\eta'}{\alpha} + \cdots\right)(1-\cos\eta')\,\mathrm{d}\eta' = \frac{1}{\alpha}(\sin\eta - \eta) + \mathcal{O}(\alpha^{-2}). \tag{35}$$

Similarly

$$\begin{aligned} J(\eta) &= -\frac{1}{\alpha}\int_0^\eta \left(1 + \frac{\eta'}{\alpha} + \frac{\eta'^2}{\alpha^2} + \cdots\right)\sin\eta'(1-\cos\eta')\,\mathrm{d}\eta' \\ &= -\frac{1}{\alpha}\left\{\frac{1}{2}(\cos\eta - 1)^2 + \frac{1}{\alpha}\left(-\eta\cos\eta + \frac{\eta}{4}\cos(2\eta)\right.\right. \\ &\quad \left.\left. + \sin\eta - \frac{\sin(2\eta)}{8}\right)\right\} + \mathcal{O}(\alpha^{-3}). \end{aligned} \tag{36}$$

Using these forms for the integrals in equations (18) and (31) and taking the limits as $E_\mathrm{s} \to 0$, we obtain

$$x(\eta) = \frac{qc}{\omega_\mathrm{l}}\gamma_0(1+\beta_0)(\sin\eta - \eta) \tag{37}$$

$$z(\eta) = \frac{c}{\omega_\mathrm{l}(1-\beta_0)}\left\{\left[\beta_0 + \frac{3}{4}q^2(1+\beta_0)\right]\eta + q^2(1+\beta_0)\left[\frac{\sin(2\eta)}{8} - \sin\eta\right]\right\}. \tag{38}$$

Equations (37) and (38) are precisely what one obtains in the absence of the uniform electric field [10, 55]. When Lorentz transformed to the frame of reference in which the electron is on average at rest, these equations produce the famous *figure 8* trajectory for the special case of an electron initially at rest ($\beta_0 = 0$) at the origin of coordinates.

Note that, prior to taking the limit as $E_\mathrm{s} \to 0$ (or $\alpha \to \infty$), we have used a series expansion in the last two terms of $z(\eta)$ similar to that employed above. Effectively,

$$\begin{aligned} z(\eta) &= \frac{c}{\omega_\mathrm{l}}\left\{\left(\frac{\omega_\mathrm{l} A_\mathrm{l}}{cE_\mathrm{s}}\right)^2\left[-\frac{1}{2\alpha}(\cos\eta - 1)^2 + \frac{1}{\alpha^2}(\eta\cos\eta - \cdots)\right.\right. \\ &\quad \left.\left. + \frac{1}{2\alpha}(\cos\eta - 1)^2\left(1 + \frac{\eta}{\alpha} + \cdots\right)\right] + \frac{1}{2}\left[\frac{\eta}{\alpha} + \frac{2\beta_0\cos\xi}{1-\beta_0\cos\xi}\right]\left(\frac{\eta}{1-\eta/\alpha}\right)\right\} \\ &= \frac{c}{\omega_\mathrm{l}}\left\{\left(\frac{\omega_\mathrm{l} A_\mathrm{l}}{cE_\mathrm{s}}\right)^2\left[\frac{1}{\alpha^2}(\eta\cos\eta - \cdots) + \frac{1}{2\alpha}(\cos\eta - 1)^2\left(\frac{\eta}{\alpha} + \cdots\right)\right]\right. \\ &\quad \left. + \frac{1}{2}\left[\frac{\eta}{\alpha} + \frac{2\beta_0\cos\xi}{1-\beta_0\cos\xi}\right]\left(\frac{\eta}{1-\eta/\alpha}\right)\right\}. \end{aligned} \tag{39}$$

This eliminates all terms involving inverse powers of E_s from $z(\eta)$. Upon close inspection one concludes that the dependence upon the uniform electric field of the electron's drift (z-motion) is through terms of ascending powers of E_s, the first occurrence being E_s, the second E_s^2 and so on. On the other hand, the dependence upon the laser electric field $E_\mathrm{l} = \omega A_\mathrm{l}/c$ is only quadratic.

Furthermore, in the limit of $E_\mathrm{s} \to 0$ (or $\alpha \to \infty$) the energy equation becomes

$$\gamma(\eta) = \gamma_0\left\{1 + \frac{q^2}{2}(1+\beta_0)\left[\cos\eta - 1\right]^2\right\}. \tag{40}$$

Equation (40) has also been arrived at before [10, 55].

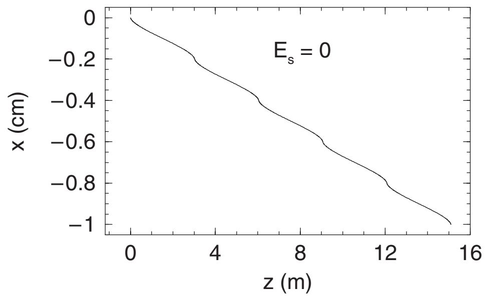

Figure 2. Electron trajectory in a linearly polarized plane-wave laser pulse, of wavelength $\lambda = 1$ μm and intensity $\approx 10^{20}$ W cm^{-2} corresponding to $q = 10$, and no uniform electric field. The trajectory has been calculated in a pulse containing five field cycles and for an electron injected initially parallel to the field ($\xi = 0$) with 50 MeV kinetic energy ($\gamma_0 = 100$).

2.3. *The $q = 0$ limit*

Another interesting limit is that of total absence of the laser field. In this case, and assuming forward initial injection ($\xi = 0$), motion of the electron is confined to a straight line, the z-axis, and its position and energy are both increasing functions of the time. This may be seen by taking the limit $q \to 0$ in equations (7)–(10) and integrating the resulting expressions subject to the same initial conditions as expressed by equation (15). The results of doing so are

$$z(t) = \frac{\gamma_0 mc^2}{eE_s}\left\{-1 + \sqrt{1 + \frac{2eE_s}{\gamma_0 mc}\left[\beta_0 t + \frac{eE_s}{2\gamma_0 mc}t^2\right]}\right\} \tag{41}$$

$$\gamma(t) = \gamma_0\sqrt{1 + \frac{2eE_s}{\gamma_0 mc}\left[\beta_0 t + \frac{eE_s}{2\gamma_0 mc}t^2\right]}. \tag{42}$$

When the limit $q \to 0$ is taken in equations (31) and (33) and after some lengthy algebraic manipulation of the resulting expressions, equations (41) and (42) follow, respectively.

2.4. *Discussion*

In order to understand the effect of the added uniform electric field of strength E_s on the motion of the electron, we have calculated some trajectories in the absence of E_s as well as in the presence of a weak E_s (by comparison to E_l). In figure 2 a typical laboratory trajectory, in the absence of E_s [55], is shown for an initially moving electron. The addition of a weak electric field has been found, both purely numerically and on the basis of our equations, not to deviate appreciably from the zero-field case. However, the situation changes quite drastically when the value of E_s approaches some critical value. More on this will be encountered in the next section.

Note that in trying to roughly estimate a velocity for the electron from the information displayed in figure 2, it is necessary (unlike in the *non-relativistic* case) to calculate the elapsed time in the laboratory from the Lorentz-invariant relation $\eta = \omega_l t - (\omega_l/c)z$. This immediately gives $t = \eta/\omega_l + z/c$. For an N cycle pulse ($\eta = 2\pi N$, $\omega_l = 2\pi c/\lambda$) one has $t = N\lambda/c + z/c$, and hence $v \approx z/t = cz/(z + N\lambda) < c$.

3. Particle acceleration

3.1. *Mechanism*

In the absence of E_s, the domain of validity of equation (40), the electron energy oscillates with η. Moreover, at the end of interaction with a pulse containing N field cycles, $\gamma(2\pi N) = \gamma_0$. In other words, the electron deposits all its energy gain back into the field as it is left behind

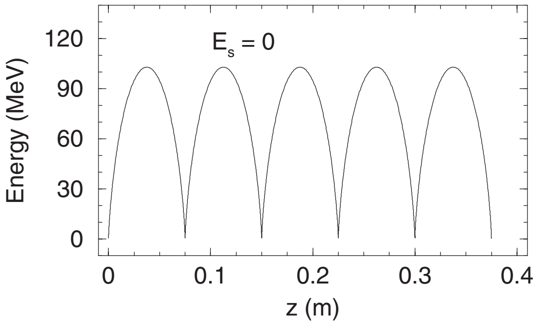

Figure 3. Electron energy versus its z coordinate in a linearly polarized plane-wave laser pulse, of wavelength $\lambda = 1\ \mu$m and intensity corresponding to $q = 10$, and no uniform electric field. The pulse has five field cycles and the electron is initially at rest at the origin of coordinates ($\gamma_0 = 1$).

the laser pulse, in agreement with the Lawson–Woodward theorem [43]. It will, however, retain some or all of that energy gain if it is extracted from the interaction region at points that correspond to $\eta \neq 2\pi N$. The gain is always a maximum if the electron is made to leave at points for which $\eta = s\pi$, where s is an odd integer.

One would want equation (33) to tell a different story: due to the secular terms the energy grows quadratically with η, a hint that may be taken to mean that a net energy gain by the electron from interaction with the radiation field is always possible. Closer inspection of equation (33), however, reveals that the terms responsible for such a gain are independent of the laser field intensity; η/α is independent of the laser frequency, too. Thus the argument of the previous paragraph regarding the gain in energy after interaction with a certain number of field cycles, or fractions thereof, is modified as follows. The electron will be accelerated continuously by the static electric field. It will also exchange tremendous amounts of energy with the radiation field. All or part of the energy absorbed from the radiation field will be retained, depending upon the value of η that corresponds to the point at which the electron is extracted from the interaction region. The gain will be a maximum if the electron is ejected after it has interacted with exactly an odd number of laser field half-cycles. Moreover, at the end of each field cycle the electron retains a small part of the energy gain in contrast to the zero-electric-field case, due to the presence of the terms quadratic and linear in η. Most of the features pointed out in this and the previous paragraphs are exhibited in figure 3.

Inspection of the denominator of the second term in equation (33) quickly suggests that if the initial electron speed or, equivalently, its initial injection energy, the applied electric field strength E_s and the frequency of the laser field all could be chosen in such a way as to make the value of α approach $2\pi N$, where N is the number of field cycles in the pulse, then the net energy gain would be large. It also suggests that this gain would be arbitrarily large if the value of α could be made arbitrarily close to $2\pi N$. Unfortunately, this would require E_s to be large and ω_l to be small. Moreover, under these circumstances, interaction would take place over a long distance, as may readily be seen from equation (31), especially for $q \gg 1$.

In other words, corresponding to every *plane-wave* laser pulse (with a definite frequency ω_l and number N of field cycles) and any given initial electron energy $\gamma_0 mc^2$, there exists a critical static electric field E_s^c given by

$$E_s^c = \frac{m\omega_l c}{2\pi N e}\gamma_0(1 - \beta_0 \cos\xi). \tag{43}$$

For values of E_s approaching E_s^c, the electron will be accelerated to high energies. When a focused laser beam is used *alone* to accelerate electrons in vacuum to high energy the phase velocity v_{ph} of the laser electric field, which is mainly responsible for the energy gain, if any, becomes greater than the speed of light. Let v_z be the forward electron speed at any one point. With $v_z < c < v_{\text{ph}}$ the electron slips quickly behind the laser pulse and decelerates. If interaction takes place over many field cycles, then the accelerating and decelerating regions almost cancel out and the electron retains virtually no net energy gain. This is the central point in what is known as the Lawson–Woodward theorem (for a thorough discussion of this theorem and its implications, see [43]).

However, the theorem does not prevent absorption of energy by the electron from the laser field if an additional static electric (or other) field is present. In the idealized plane-wave description, $v_{\text{ph}} = c$. Under action of the *right* added uniform electric field, E_{s}^{c}, the electron stays approximately in phase with the field responsible for the acceleration mechanism (the laser) as long as $v_z \approx c$. This may further be elucidated by the following crude argument. When the (nonrelativistic) Newton equation of motion of the electron in the uniform electric field E_{s} alone is integrated, it gives

$$v_z \approx \frac{eE_{\text{s}}}{m}t \to \frac{mc}{eE_{\text{s}}} \approx \frac{ct}{v_z}. \tag{44}$$

On the other hand, for parallel injection ($\xi = 0$) equation (43) above may be written as

$$\begin{aligned} \frac{mc}{eE_{\text{s}}} &= \frac{2\pi N}{\omega_0\gamma_0(1-\beta_0)} \\ &= \frac{N\tilde{\lambda}}{c} \end{aligned} \tag{45}$$

where $\tilde{\lambda} = \lambda\sqrt{(1+\beta_0)/(1-\beta_0)}$ is the Doppler-shifted laser wavelength as seen by the electron *initially*. Within the context of this crude analysis, we may regard $\tilde{\lambda}$ as being approximately a constant. Thus, equating the right-hand sides of equations (44) and (45) we obtain

$$N\tilde{\lambda} = \frac{c^2t}{v_z} \approx v_z t. \tag{46}$$

In the second line of equation (46) we have made the replacements $v_{\text{ph}} = c \approx v_z$. Therefore, what this equation simply says is that E_{s} causes the electron to move at $v_z \approx v_{\text{ph}}$, so that after an interaction time t it is still *riding* in phase with the accelerating electric field of the laser. Thus it continues to absorb energy from the radiation field.

This will now be illustrated by an example. To accelerate an electron from rest to several hundred GeV, employing a 50-cycle laser pulse of wavelength 1 μm and intensity corresponding to $q = 100$ (intensity $\approx 10^{22}$ W cm^{-2}), requires a critical electric field of strength in the neighbourhood of $E_{\text{s}}^{\text{c}} = 1.02487 \times 10^{10}$ V m^{-1}. Plasma-based accelerators [43] are capable of sustaining electric field strengths far in excess of this value of E_{s}^{c}. When a field of strength $E_{\text{s}} = 10^{10}$ V m^{-1} is used, our equations yield the results displayed in figure 4. Note from figure 4(*a*) that interaction between the field and the electron terminates in about 180 ns, and takes place over a distance of approximately 54 m, as may be inferred from figure 4(*b*). The energy gain is shown in figure 4(*c*) where a cycle-by-cycle buildup is clearly visible. Note, for example, that approximately between the 20 and 44 m marks on the z-axis, the energy of the electron is already over about 0.5 TeV. In other words, if the electron is extracted anywhere between these two points its energy will be in the range of roughly 0.5–0.7 TeV. If extracted at $z \approx 30$ m, for example, the electron energy gradient (defined here as the exit energy divided by the total axial distance of travel) will be about 23 GeV m^{-1}.

On the other hand, in the absence of the uniform electric field, $E_{\text{s}} = 0$, the maximum energy gain possible from the laser field would be about 10 GeV, as may be seen in figure 5(*b*). Alternatively, if the (huge) electric field were to act on the electron alone (as in a conventional

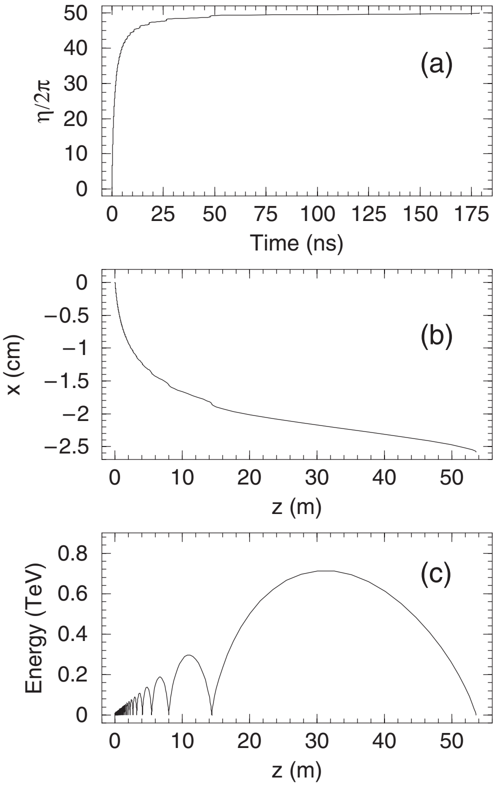

Figure 4. For an electron initially at rest at the origin of coordinates and subsequently subjected to a plane-wave laser pulse of wavelength $\lambda = 1\ \mu$m and intensity $\approx 10^{22}$ W cm^{-2} corresponding to $q = 100$, and a uniform electric field of strength $E_s = 10^{10}$ V m^{-1}, the following are shown: (*a*) the phase of the laser field in units of 2π versus the time in nanoseconds; (*b*) the electron trajectory in the xz plane and (*c*) the electron energy versus its z coordinate.

linear accelerator) over the same time interval (or, equivalently, over the same spatial distance of 30 m) the electron will reach an energy of only 0.3 TeV, as may be seen in figure 5(*c*). This corresponds to an energy gradient of 10 GeV m^{-1}. Thus one arrives at the conclusion that the dc electric field and the laser field lead to more electron energy gain per metre when acting together than if each of them were to act on the electron alone.

Note the behaviour of the laser field phase as a function of the time through

$$t = \eta/\omega_l + z(\eta)/c. \tag{47}$$

In the absence of the static electric field, the forward drift Δz, as a result of interaction with any full cycle of the laser field, is independent of η. For n an integer equation (38) gives

$$\begin{aligned}\Delta z &\equiv z[2\pi(n+1)] - z[2\pi n] \\ &= \frac{2\pi c}{\omega_l(1-\beta_0)}\left[\beta_0 + \frac{3}{4}q^2(1+\beta_0)\right].\end{aligned} \tag{48}$$

Equation (48) explains the staircase structure of figure 5(*a*), which sharply contrasts with figure 4(*a*). Figure 4(*a*) has been calculated on the basis of equation (47) in conjunction with equation (31). In this case, the dependence of Δz upon η and E_s is very strong and results in narrower and higher steps during interaction with most of the field cycles, save for the last few. During interaction with the last few field cycles, the speed of the electron will be close to that of light and it will almost be carried along with the laser wave.

Other calculations (not shown here) led to the conclusion that an energy gain of over 1 TeV may be reached when the number of field cycles in the pulse is doubled. In this case both E_s^c

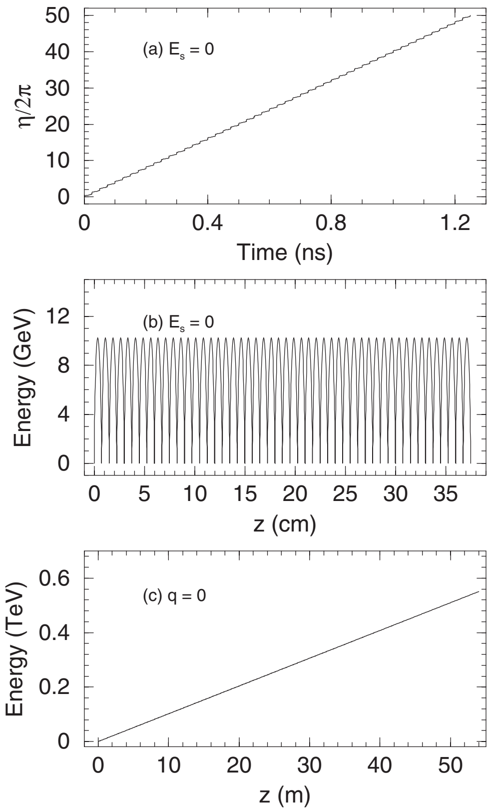

Figure 5. For electron and laser field parameters similar to those of figure 4, and *no* uniform electric field, the following are shown: (*a*) the phase of the laser field in units of 2π versus the time in nanoseconds; (*b*) the electron energy versus its z coordinate. In (*c*) we show the electron energy in the presence of a uniform electric field of strength $E_s = 10^{10}$ V m^{-1} *acting alone* over a time interval equal to the full interaction time encountered in figure 4.

and E_s are halved. However, there is a price for this, namely, the energy gain will take place over a longer distance.

Consider the situation when injection is made at a finite angle ξ to the laser propagation direction. In order to obtain a sizeable energy gain the critical dc field strength, as dictated by equation (43), must be approached. Parallel injection ($\xi = 0$) corresponds to a minimum E_s^c for a given set of laser and electron parameters; the decision to inject at a nonzero ξ calls for a higher E_s^c. Figure 6 has been produced for three injection angles. It appears from the figure that a higher average energy gradient may be achieved by injection at an angle than in the parallel case.

3.2. Radiation losses

The accelerated electron emits radiation and thus loses energy. This so-called *radiation loss* places a severe limit on the maximum energy attainable by the particle in conventional circular accelerators. To estimate the radiation losses by the electron with whose dynamics we have been concerned thus far, we employ the relativistic generalization of the Larmor formula [57] for the total instantaneous power

$$P(t) = \frac{2}{3}\frac{e^2}{c}\gamma^6 \left\{ \left[\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}t}\right]^2 - \left[\boldsymbol{\beta} \times \frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}t}\right]^2 \right\}. \tag{49}$$

For use in this expression and in order to gain more insight into the dynamics of the electron, we need expressions for the (scaled) velocity components as functions of the phase of the laser

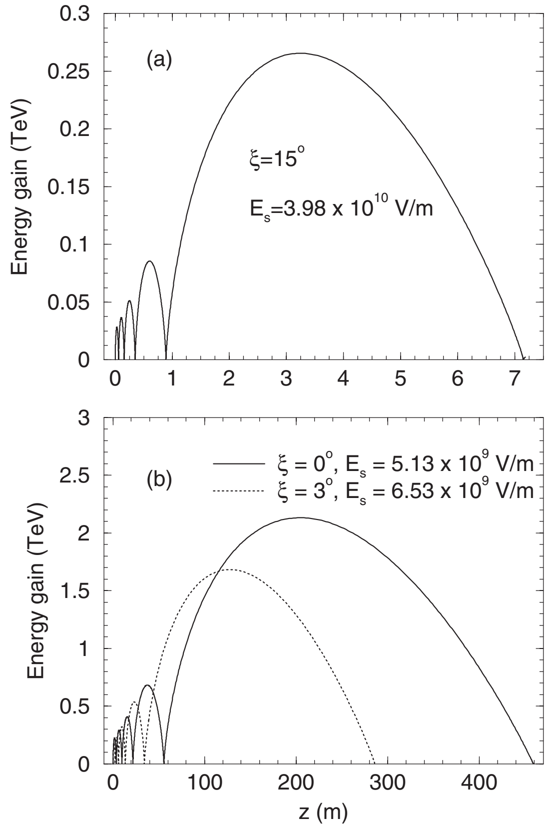

Figure 6. Energy gain versus forward distance of travel of an electron initially injected with energy close to 4.5 MeV ($\gamma_0 = 10$) into a laser field whose parameters are similar to those of figure 4 (except for the number of field cycles; here $N = 5$) in addition to the dc electric field E_s. Note that the critical electric fields are $E_s^c = 3.9884 \times 10^{10}$, $5.137\,25 \times 10^9$, and $6.534\,77 \times 10^9$ V m^{-1}, respectively.

field. Equations (13), (16), (17) and (33) yield ($\xi = 0$)

$$\beta_x(\eta) = \frac{q(\cos\eta - 1)}{\gamma(\eta)} \tag{50}$$

$$\beta_y(\eta) = 0 \tag{51}$$

$$\beta_z(\eta) = 1 - \frac{\gamma_0(1-\beta_0)(1-\eta/\alpha)}{\gamma(\eta)}. \tag{52}$$

These equations will also be useful for the calculation of the radiation spectra to be performed in section 4. Plots of β_x and β_z versus the number of field cycles are given in figure 7.

We now turn $P(t)$ into a function of the laser field phase η using the chain rule of differentiation and a well known vector identity. When equation (13) is finally used in the resulting expression, there results

$$P(\eta) = \frac{2}{3}\frac{(e\tilde{\omega}\gamma)^2}{c}\left(1 - \frac{\eta}{\alpha}\right)^2 \left\{ \left[\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}\eta}\right]^2 + \gamma^2 \left[\boldsymbol{\beta} \cdot \frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}\eta}\right]^2 \right\} \tag{53}$$

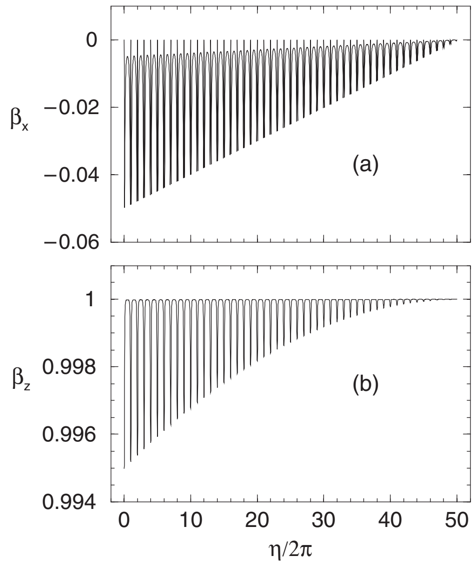

Figure 7. Components of the velocity of the electron, scaled by the speed of light, versus the number of laser field cycles. The parameters used are $q = 10$, $\lambda = 1\ \mu$m, $\gamma_0 = 10$ and $E_s = 5.1 \times 10^8$ V m^{-1}.

where

$$\tilde{\omega} = \omega_l \sqrt{\frac{1-\beta_0}{1+\beta_0}} \tag{54}$$

is the Doppler-shifted frequency of the laser field upon electron initial injection. The energy lost by emission of radiation may be compared to the energy gained at any point in the course of interaction with the radiation field by calculating the quantity $\Gamma(\eta)$ defined by

$$\Gamma(\eta) = \frac{2\pi}{\omega_l}\frac{P}{\mathcal{E}} = \frac{4\pi}{3\omega_l c}\frac{(e\tilde{\omega}\gamma)^2}{\gamma mc^2}\left(1-\frac{\eta}{\alpha}\right)^2\left\{\left[\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}\eta}\right]^2+\gamma^2\left[\boldsymbol{\beta}\cdot\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}\eta}\right]^2\right\}. \tag{55}$$

$\Gamma(\eta)$ is the ratio of the electron's radiation loss during interaction with one field cycle to its instantaneous energy. With P a varying function of η, calculation of the radiation loss over the duration of one cycle as $2\pi P/\omega_l$ is only approximate at best. Note that in the acceleration regime described above, the quantity $(1 - \eta/\alpha)$ becomes almost zero, and hence so do the radiation losses, as the electron interacts with the last few field cycles. This is shown very clearly in figure 8.

4. Emission spectra

With the trajectory equations now completely known we can embark on a discussion of the emission spectra, resulting from scattering the laser light by the electron in the presence of the static electric field, following [11]. Effectively, the discussion will be based upon the following equation, good for far-away observation points (the far-field approximation):

$$\frac{\mathrm{d}^2E(\omega,\Omega)}{\mathrm{d}\Omega\,\mathrm{d}\omega} = \frac{e^2}{4\pi^2c}\left|\int_0^T \frac{\hat{\boldsymbol{n}}\times[\hat{\boldsymbol{n}}-\boldsymbol{\beta}(t)]\times\dot{\boldsymbol{\beta}}(t)}{[1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}(t)]^2}\exp\left\{\mathrm{i}\omega\left[t-\frac{\hat{\boldsymbol{n}}\cdot\boldsymbol{r}(t)}{c}\right]\right\}\mathrm{d}t\right|^2 \tag{56}$$

where E is used here to denote the radiated energy, $\dot{\boldsymbol{\beta}}$ is the electron acceleration scaled by the speed of light, $\hat{\boldsymbol{n}}$ is a unit vector in the direction of propagation of the emitted radiation (direction of observation) and T is the time interval over which interaction between the electron

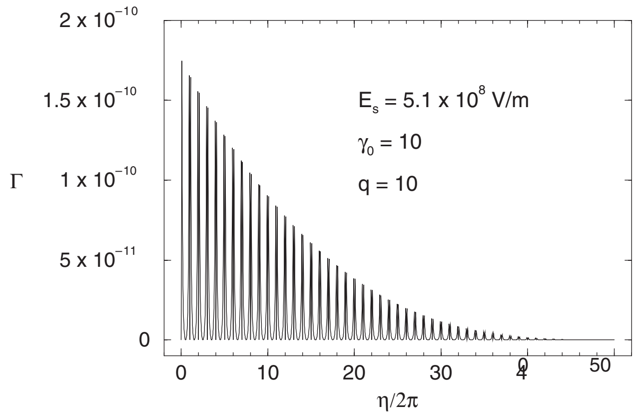

Figure 8. The ratio of the energy loss by the electron, through re-emission of radiation during interaction with one field cycle, to its instantaneous energy is shown here versus the number of field cycles $\eta/2\pi$. The radiation field has a wavelength of 1 μm.

and the laser field takes place. In what follows, we will report the spectra in terms of the doubly differential scattering cross section given by

$$\frac{\mathrm{d}^2\sigma\,(\omega,\Omega)}{\mathrm{d}\Omega\,\mathrm{d}\omega} = \frac{1}{T}\frac{8\pi c r_0^2}{(eq\omega_l)^2}\frac{\mathrm{d}^2E\,(\omega,\Omega)}{\mathrm{d}\Omega\,\mathrm{d}\omega}. \tag{57}$$

This quantity has been obtained by dividing the radiant energy, emitted into a unit solid angle per unit frequency per unit time, by the incident energy flux, $(eq\omega_l)^2/8\pi c r_0^2$, r_0 being the classical electron radius. An integration by parts may next be performed on equation (56), which when followed by a change of variable from t to η results in

$$\frac{1}{r_0^2}\frac{\mathrm{d}^2\sigma\,(\omega,\Omega)}{\mathrm{d}\Omega\,\mathrm{d}\omega} = \frac{\omega_l}{N(q\pi\omega_l)^2}\left|\boldsymbol{F}(\omega) - \mathrm{i}\left(\frac{\omega}{\omega_l}\right)\boldsymbol{G}(\omega)\right|^2 \tag{58}$$

where

$$\boldsymbol{F}(\omega) = \left[\frac{\hat{\boldsymbol{n}}\times\hat{\boldsymbol{n}}\times\boldsymbol{\beta}(\eta)}{1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}(\eta)}\right]\exp\left\{\mathrm{i}\frac{\omega}{\omega_l}\left[\eta + \frac{\omega_l}{c}\left[z(\eta) - \hat{\boldsymbol{n}}\cdot\boldsymbol{r}(\eta)\right]\right]\right\}\Bigg|_0^{2\pi N}. \tag{59}$$

$$\boldsymbol{G}(\omega) = \frac{\omega_l}{c}\int_0^{2\pi N}\left[\hat{\boldsymbol{n}}\times\hat{\boldsymbol{n}}\times\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\eta}\right]\exp\left\{\mathrm{i}\frac{\omega}{\omega_l}\left[\eta + \frac{\omega_l}{c}\left[z(\eta) - \hat{\boldsymbol{n}}\cdot\boldsymbol{r}(\eta)\right]\right]\right\}\mathrm{d}\eta. \tag{60}$$

Note that atomic units, with $e = m = 1$, have been used in equations (58)–(60). In spherical polar coordinates, $\hat{\boldsymbol{n}} = (n_1, n_2, n_3) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. As an example, let us try to work out the forward spectrum analytically. With $\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}}$, equation (59) shows that $\boldsymbol{F}(\omega) = 0$. Furthermore, from equation (60) we infer that $\boldsymbol{G}(\omega)$ has a nonvanishing component only along the x-axis with the magnitude

$$G_x(\omega) = q\gamma_0(1+\beta_0)\int_0^{2\pi N}\left[\frac{1-\cos\eta}{1-\eta/\alpha}\right]\mathrm{e}^{\mathrm{i}(\omega/\omega_l)\eta}\,\mathrm{d}\eta. \tag{61}$$

The integral in equation (61) can be performed exactly analytically but the result is quite lengthy and reporting it in this paper may not serve any particular purpose. The resulting forward spectrum, calculated using the exact analytical expression, is shown in figure 9. It consists of a single sharp peak at the fundamental frequency $\omega = \omega_l$. Other less prominent humps also exist on both sides of the fundamental. This should not be surprising since a

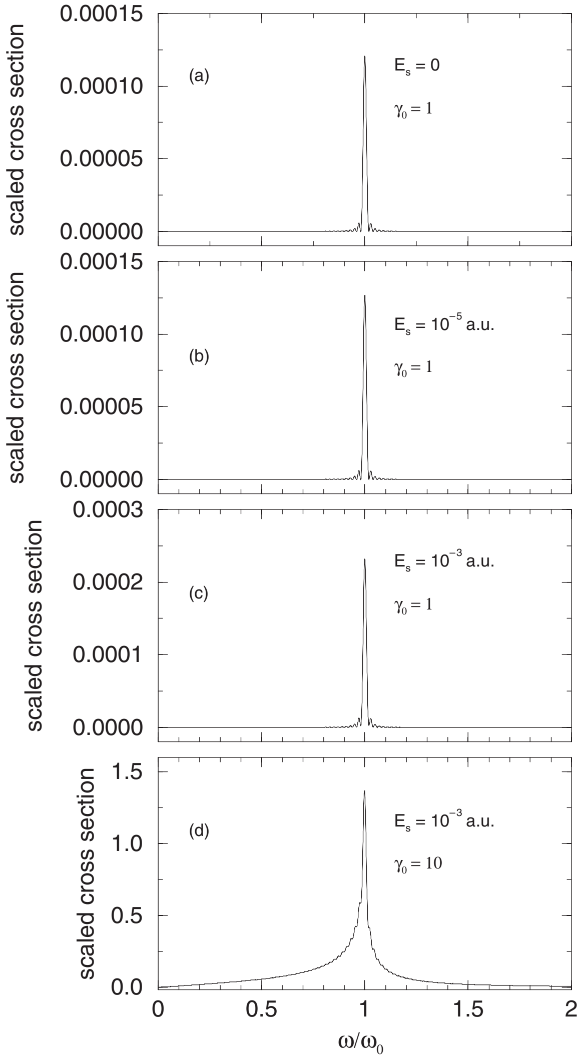

Figure 9. Frequency spectrum of the radiation, scattered by the electron in the combined presence of the laser pulse and uniform electric field, that would be observed by a detector on the z-axis and facing the origin of coordinates. The incident radiation has a wavelength of $\lambda = 1\ \mu$m.

detector, with a circular opening centred on the z-axis and facing the origin of coordinates, will not be looking straight into the (almost straight line) trajectory of the electron. Rather it would be gathering light from a small cone around the z-axis. This cone intersects the narrow cone (around the electron trajectory) of radiation given off by the relativistic electron.

Note also that when equation (61) is substituted back into equation (58) one finds out that the resulting expression for the scattering cross section is independent of the laser intensity parameter q. It is, however, proportional to the Doppler-shifted laser frequency (upon injection of the electron along the forward direction). During interaction with the fields the Doppler shift varies with η and is a function of the uniform electric field strength E_s. Hence, the frequency modulation effects present in the figures should come as no surprise. Such effects have been recently investigated by Hartemann *et al* [55] in the backscattered spectrum and in the absence of the axial electric field. Furthermore, the dependence upon E_s in equation (61) is weak. It is all in the denominator of the integrand, which affects the value of G_x, and hence the spectrum, only in the acceleration regime described above. Note also the broadening and left–right asymmetry in figure 8(d). We conclude by noting that spectra corresponding to other observation directions may, in principle, be calculated numerically on the basis of equations (58)–(60). However, the integrands in equations (59) and (60) are highly oscillating functions of η. Thus numerical results based on them should be viewed with caution [11, 12].

5. Summary and conclusions

We have solved exactly analytically the relativistic equations of motion for a single electron in vacuum in the presence of a linearly polarized plane-wave laser field of arbitrary intensity in addition to a uniform static electric field oriented anti-parallel to the laser direction of propagation. We have demonstrated that when the strength of the added dc electric field approaches a critical value that depends on the laser frequency, number of field cycles and initial forward electron speed, conditions suitable for particle acceleration are achieved, especially for electrons produced almost at rest as perhaps via an ionization process. We have also shown that the emission of radiation is negligible compared to the electron energy gain in this acceleration regime and that the radiation emitted along the particle's initial direction of motion (parallel injection) consists only of the laser fundamental frequency.

We would like to stress that our effort in this paper has been aimed at assessing the possibility of using a dc electric and laser field combination as a means of accelerating electrons to high energies. Although such a scheme may not be feasible by today's technology, yet it may not be ruled out altogether by near-future innovations. This applies with equal weight to both laser field intensity and dc electric field strength requirements, especially when it comes to maintaining either requirement over a distance of metres.

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